Lecture 10 - geometric invariant theory

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Main goal: cook up something close to a quotient space

$\mathcal{X} \longrightarrow X$ universal for maps to alg. spaces

Definition 4.1. We say that $\phi : \mathcal{X} \to Y$ is a *good moduli space* if the following properties are satisfied:

(i) ϕ is cohomologically affine.

(ii) The natural map $\mathcal{O}_Y \xrightarrow{\sim} \phi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Simple looking definition, main properties

Main Properties. If $\phi : \mathcal{X} \to Y$ is a good moduli space, then:

(1) ϕ is surjective and universally closed (in particular, Y has the quotient topology).

(2) Two geometric points x_1 and $x_2 \in \mathcal{X}(k)$ are identified in Y if and only if their closures $\overline{\{x_1\}}$ and $\overline{\{x_2\}}$ in $\mathcal{X} \times_{\mathbb{Z}} k$ intersect.

(3) If $Y' \to Y$ is any morphism of algebraic spaces, then $\phi_{Y'} : \mathcal{X} \times_Y Y' \to Y'$ is a good moduli space.

(4) If $\mathcal{X}$ is locally noetherian, then ϕ is universal for maps to algebraic spaces.

(5) If $\mathcal{X}$ is finite type over an excellent scheme S, then Y is finite type over S.

(6) If $\mathcal{X}$ is locally noetherian, a vector bundle $\mathcal{F}$ on $\mathcal{X}$ is the pullback of a vector bundle on Y if and only if for every geometric point $x : \operatorname{Spec} k \to \mathcal{X}$ with closed image, the G_x-representation $\mathcal{F} \otimes k$ is trivial.

(Alper '08)

The main example is G linearly reductive, R affine

$$\operatorname{Spec}(R)/G \longrightarrow \operatorname{Spec}(R^G)$$

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In fact, this is a local model in great generality

E.g. quotient stack whose good moduli space is a scheme $\Longrightarrow$ locally of this form

Theorem 1.2. *Let $\mathcal{X}$ be a quasi-separated algebraic stack, locally of finite type over an algebraically closed field k, with affine stabilizers. Let $x \in \mathcal{X}(k)$ be a point and $H \subseteq G_x$ be a subgroup scheme of the stabilizer such that H is linearly reductive and G_x/H is smooth (resp. étale). Then there exists an affine scheme $\mathrm{Spec}\, A$ with an action of H, a k-point $w \in \mathrm{Spec}\, A$ fixed by H, and a smooth (resp. étale) morphism*

$$f\colon ([\mathrm{Spec}\, A/H], w) \to (\mathcal{X}, x)$$

such that $BH \cong f^{-1}(BG_x)$; in particular, f induces the given inclusion $H \to G_x$ on stabilizer group schemes at w. In addition, if $\mathcal{X}$ has affine diagonal, then the morphism f can be arranged to be affine.

Theorem 2.9. *Let $\mathcal{X}$ be a locally noetherian algebraic stack over k. Suppose there exists a good moduli space X such that the moduli map $\pi\colon \mathcal{X} \to X$ is of finite type with affine diagonal. If $x \in \mathcal{X}(k)$ is a closed point, then there exists an affine scheme $\mathrm{Spec}\, A$ with an action of G_x and a cartesian diagram*

$$\begin{tikzcd}
[\mathrm{Spec}\, A/G_x] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
\mathrm{Spec}\, A/\!\!/G_x \arrow[r] & X
\end{tikzcd}$$

such that $\mathrm{Spec}\, A/\!\!/G_x \to X$ is an étale neighborhood of $\pi(x)$.

This leads to a strategy for constructing spaces

$\rightsquigarrow$ Cover by affine quotient stacks!